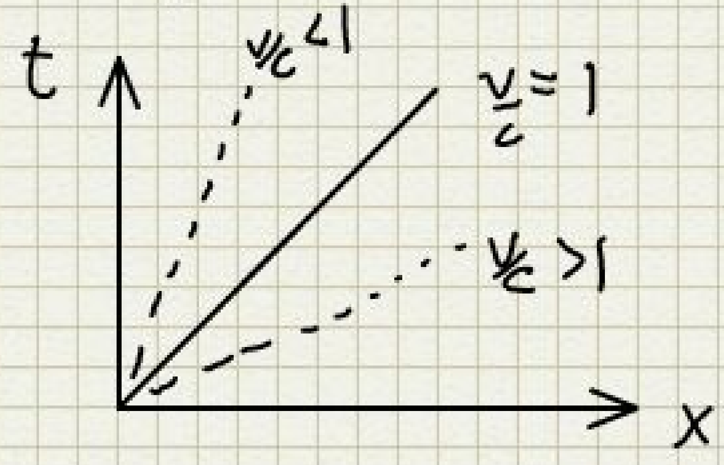
Lecture 1. Special relativity

- existence of inertial coordinate system (not accelerated)
- equivalence of the inertial frames (principle of relativity)
- universality of the speed of light

4-d space-time



$$\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$$ ← interval in flat spacetime Δ ← are can be arbitrary large.

Δs^2 - invariant under coordinate transformation (observer) could be negative (timelike), positive (spacelike), or zero (null).

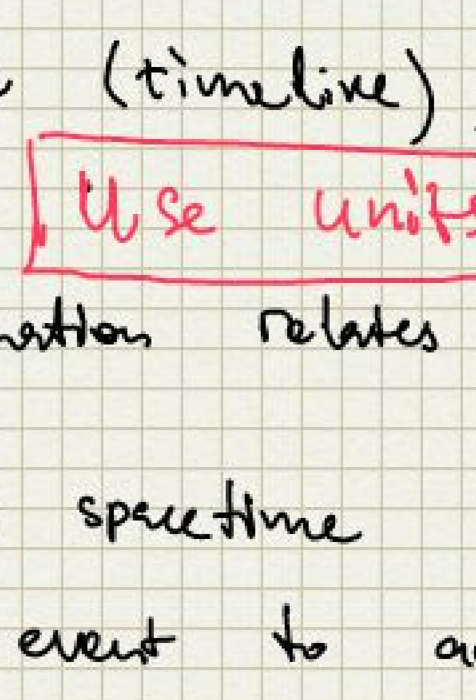
$\boxed{\text{Use units } G = c = 1}$

Lorentz transformation relates inertial coordinate frames.

Event : point in spacetime $M(t, x, y, z)$, Vector points from one event to another : $\Delta x^\mu : \{\Delta t, \Delta x, \Delta y, \Delta z\}$

$\mu \to$ coordinate index : $\mu = 0,1,2,3$: $\Delta x^0 = \Delta t$, $\Delta x^1 = \Delta x$, $\Delta x^2 = \Delta y$, $\Delta x^3 = \Delta z$

four velocity



u^μ - vector tangent to the trajectory (world line)

$$u^\mu = \frac{dx^\mu}{dt}$$

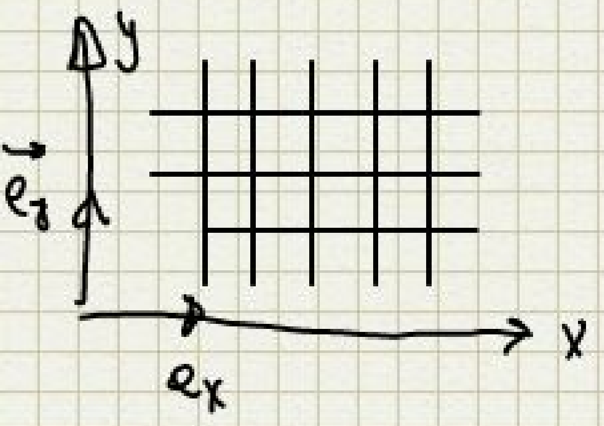
in momentary comoving frame : $u^\mu : \{1, 0, 0, 0\}$

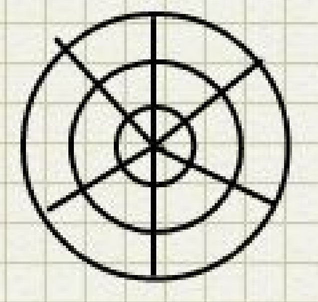
4 - momentum $\quad p^\mu = m u^\mu \to \{E, \vec{p}\}$

Lorentz transformation !

Curvelinear coordinates

Consider two coordinate frames (i) Cartesian (ii)





r, θ

x, y

$$r = \sqrt{x^2 + y^2} \quad ; \quad \theta = \arctan \frac{y}{x} \quad \Big| \quad x = r\cos\theta, \; y = r\sin\theta$$

Consider a small increment:

$$\begin{cases} \Delta r = \dfrac{\partial r}{\partial x}\Delta x + \dfrac{\partial r}{\partial y}\Delta y = \cos\theta\,\Delta x + \sin\theta\,\Delta y \\ \Delta\theta = \dfrac{\partial \theta}{\partial x}\Delta x + \dfrac{\partial \theta}{\partial y}\Delta y = -\dfrac{\sin\theta}{r}\Delta x + \dfrac{\cos\theta}{r}\Delta y \end{cases}$$

General coordinate transformation:

$$\Delta \xi^i = \frac{\partial \xi^i}{\partial x^j}\Delta x^j \quad \text{(sum over repeated indices)}$$

$$\equiv \sum_j \frac{\partial \xi^i}{\partial x^j}\Delta x^j$$

In our case $\xi^i = \{r, \theta)$; $x^j : \{x, y\}$.

$\Lambda^{\bar{\alpha}}{}_{\beta}$ ← matrix of coordinate transformation

$\Delta\xi^i \to$ 2-d vector $\Rightarrow$ Consider a general vector $\vec{V}$

$V^{\bar{\alpha}} \to$ components of a vector in $\xi^{\bar{\alpha}}$ coordinates

$V^{\beta} \to$ components of a vector in x^{β} coordinates

($\alpha, \beta \to$ dummy indices, running 0,1,2,3)

$$\boxed{V^{\bar{\alpha}} = \Lambda^{\bar{\alpha}}{}_{\beta} V^{\beta}} \quad ; \quad \Lambda^{\bar{\alpha}}{}_{\beta} = \frac{\partial \xi^{\bar{\alpha}}}{\partial x^{\beta}} \cdot \left(\text{tensor} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$$

Consider a scalar field:

$\phi(\xi^{\bar{\alpha}})$

Take total derivative: $d\phi = \frac{\partial \phi}{\partial \xi^{\bar{\alpha}}} d\xi^{\bar{\alpha}}$

if we assume $\xi^{\bar{\alpha}} = \xi^{\bar{\alpha}}(x^{\beta}) \rightarrow x^{\alpha} = x^{\alpha}(\xi^{\bar{\beta}})$

$\rightarrow \frac{d\phi}{\partial \xi^{\bar{\alpha}}} = \frac{\partial \phi}{\partial x^{\beta}} \frac{\partial x^{\beta}}{\partial \xi^{\bar{\alpha}}} \equiv \Lambda^{\beta}{}_{\bar{\alpha}} \frac{\partial \phi}{\partial x^{\beta}}$

$\rightarrow \boxed{d\phi_{\bar{\alpha}} = \Lambda^{\beta}{}_{\bar{\alpha}}\, d\phi_{\beta}}$ tensor $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Consider basis vectors:

$\vec{e}_x, \vec{e}_y$: $\{1, 0\}, \{0, 1\}$ ← in cartesian coord.

$\vec{e}_{\bar{\alpha}} = \Lambda^{\beta}{}_{\bar{\alpha}} \vec{e}_{\beta}$ $\bar{\alpha} \rightarrow$ polar, $\beta \rightarrow$ cartesian

$\vec{e}_r = \frac{\partial x}{\partial r} \vec{e}_x + \frac{\partial y}{\partial r} \vec{e}_y = \cos\theta\, \vec{e}_x + \sin\theta\, \vec{e}_y$

$\vec{e}_\theta = \frac{\partial x}{\partial r} \vec{e}_x + \frac{\partial y}{\partial r} \vec{e}_y = -r\sin\theta\, \vec{e}_x + r\cos\theta\, \vec{e}_y$

Metric tensor ⇒ defines the inner (dot) product

$g(\vec{e}_\alpha, \vec{e}_\beta)$: in cartesian frame $g(\vec{e}_\alpha, \vec{e}_\beta) = \delta_{\alpha\beta}$

$\delta_{\alpha\beta}$ - Kronecker delta : $\vec{e}_x \cdot \vec{e}_x = 1$, $\vec{e}_y \cdot \vec{e}_y = 1$, $\vec{e}_x \cdot \vec{e}_y = 0$

$\delta_{\alpha\beta} = 1$ if $\alpha = \beta$.

$g_{\alpha\beta} = \vec{e}_\alpha \cdot \vec{e}_\beta \rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

In polar coord $g_{\bar{\alpha}\bar{\beta}} = \vec{e}_{\bar{\alpha}} \cdot \vec{e}_{\bar{\beta}} = \dots \overset{\text{(see above)}}{=} \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$

f-n of a point (position)

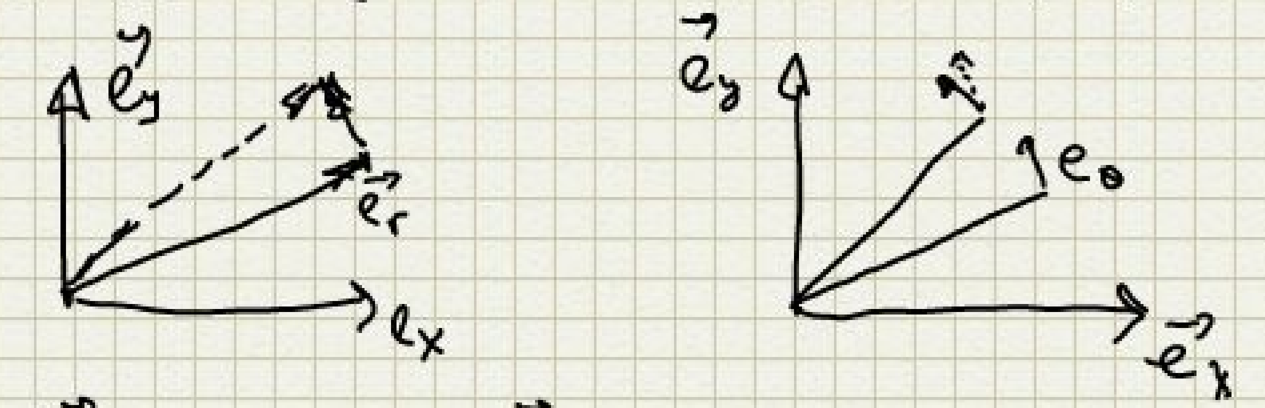
Interval: $ds^2 = dx^2 + dy^2 = |dr\, \vec{e}_r + d\theta\, \vec{e}_\theta|^2 = dr^2 + r^2 d\theta^2$

$$\|d\ell\|^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = g_{\bar\alpha\bar\beta}\, dx^{\bar\alpha} dx^{\bar\beta}$$

inverse : $g^{\alpha\beta} g_{\beta\nu} = \delta^\alpha_\nu$

$$\rightarrow g^{\bar\alpha\bar\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$$

Consider derivatives of a basis vector



$$\frac{\partial \vec{e}_r}{\partial r} = 0 \ ; \quad \frac{\partial \vec{e}_r}{\partial \theta} = -\sin\theta\, \vec{e}_x + \cos\theta\, \vec{e}_y = \frac{1}{r} \vec{e}_\theta$$

$$\frac{\partial \vec{e}_\theta}{\partial r} = \frac{1}{r} \vec{e}_\theta \qquad \frac{\partial \vec{e}_\theta}{\partial \theta} = -r\, \vec{e}_\theta$$

Consider a vector field : $\vec{V}(\xi^k)$

$$\frac{\partial \vec{V}}{\partial r} = \frac{\partial}{\partial r}\left[V^r \vec{e}_r + V^\theta \vec{e}_\theta \right] = \frac{\partial V^r}{\partial r} \vec{e}_r + V_r \frac{\partial \vec{e}_r}{\partial r} +$$

$$+ \frac{\partial V^\theta}{\partial r} \vec{e}_\theta + V^\theta \frac{\partial \vec{e}_\theta}{\partial r} \equiv \frac{\partial V^{\bar\alpha}}{\partial x} \vec{e}_{\bar\alpha} + V^{\bar\alpha} \frac{\partial \vec{e}_{\bar\alpha}}{\partial r}$$

$$\Rightarrow \boxed{\frac{\partial V}{\partial \xi^{\bar\beta}} = \frac{\partial V^{\bar\alpha}}{\partial \xi^{\bar\beta}} \vec{e}_{\bar\alpha} + V^{\bar\alpha} \frac{\partial \vec{e}_{\bar\alpha}}{\partial \xi^{\bar\beta}}}$$

It's vector → can be decomposed in a basis :

$$\frac{\partial \vec{e}_{\bar\alpha}}{\partial \xi^{\bar\beta}} = \Gamma^{\bar\mu}_{\bar\alpha\bar\beta}\, \vec{e}_{\bar\mu}$$

Christoffel symbols

$\Gamma^{\bar\mu}_{\bar\alpha\bar\beta} = \Gamma^{\bar\mu}_{\bar\beta\bar\alpha}$ ← doesn't transform as tensor in polar coordinates:

$$\Gamma^{\theta}_{r\theta} = \Gamma^{\theta}_{\theta r} = \frac{1}{r} \; ; \; \Gamma^{\theta}_{\theta\theta} = \Gamma^{\mu}_{rr} = 0$$

$$\frac{\partial \vec{V}}{\partial \xi^{\bar{\beta}}} = \frac{\partial V^{\bar{\alpha}}}{\partial \xi^{\bar{\beta}}} \vec{e}_{\bar{\alpha}} + V^{\bar{\alpha}} \Gamma^{\bar{\mu}}_{\bar{\alpha}\bar{\beta}} \vec{e}_{\bar{\mu}}$$ ← true in any coordinate frame

in Cartesian frame $\Gamma^{\alpha}_{\mu\nu} = 0$

$$\frac{\partial \vec{V}}{\partial x^{\beta}} = \left(\frac{\partial V^{\alpha}}{\partial x^{\beta}} + V^{\mu} \Gamma^{\alpha}_{\mu\beta} \right) \vec{e}_{\alpha} .$$

Introduce: $V^{\alpha}{}_{;\beta} \equiv V^{\alpha}{}_{,\beta} + V^{\mu} \Gamma^{\alpha}_{\mu\beta}$

(;) → covariant derivative (,) partial derivative

$$V^{\alpha}{}_{,\beta} \equiv \frac{\partial V^{\alpha}}{\partial x^{\beta}} .$$

Covariant deriv. ⇒ takes into account change in the basis vectors as we move from one point to another.

In cartesian coord $V^{\alpha}{}_{;\beta} = V^{\alpha}{}_{,\beta}$.

$V^{\alpha}{}_{;\beta}$ → tensor $\binom{1}{1}$

In other coord. frame:

$$V^{\bar{\alpha}}{}_{;\bar{\beta}} = V^{\bar{\alpha}}{}_{,\bar{\beta}} + \Gamma^{\bar{\alpha}}_{\bar{\mu}\bar{\beta}} V^{\bar{\mu}}$$ or

$$V^{\bar{\alpha}}{}_{;\bar{\beta}} = \Lambda^{\bar{\alpha}}{}_{\mu} \Lambda^{\nu}{}_{\bar{\beta}} V^{\mu}{}_{;\nu} .$$

Laplacian $\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} \Rightarrow \varphi^{,\alpha}{}_{,\alpha}$

Generalizing $\varphi^{;\alpha}{}_{;\alpha}$

$\varphi^{;\alpha} \equiv V^\alpha$ – vector $\rightarrow V^\alpha{}_{;\alpha} = V^\alpha{}_{,\alpha} + \Gamma^\alpha_{\beta\alpha} V^\beta \rightarrow$

$\varphi^{;\alpha}{}_{;\alpha} = \varphi^{,\alpha}{}_{,\alpha} + \Gamma^\alpha_{\beta\alpha}\varphi^{,\beta}$; $\Gamma^\alpha_{\beta\alpha} \Rightarrow \Gamma^r_{\beta r} + \Gamma^\theta_{\beta\theta}$

$\rightarrow \Gamma^\alpha_{\beta\alpha} = \begin{cases} \frac{1}{r} & \beta = r \\ 0 & \beta = \theta \end{cases} \rightarrow$

$\Delta\varphi = \varphi^{,r}{}_{,r} + \varphi^{,\theta}{}_{,\theta} + \frac{1}{r}\varphi^{,r} \rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\,\varphi^{,r}\right) + \varphi^{,\theta}{}_{,\theta}$

Metric is use to rise & lower the indices

$V^\alpha = g^{\alpha\beta} V_\beta$; $V_\beta = g_{\alpha\beta} V^\alpha$; $g_{\alpha\beta} = g_{\beta\alpha}$.

map between $\binom{1}{0} \Leftrightarrow \binom{0}{1}$

$\varphi^{,r} = g^{rr}\varphi_{,r} + g^{r\theta}\varphi_{,\theta} = \varphi_{,r} = \frac{\partial\varphi}{\partial r}$

$\varphi^{,\theta} = g^{\theta r}\varphi_{,r} + g^{\theta\theta}\varphi_{,\theta} = g^{\theta\theta}\varphi_{,\theta} = \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta}$

$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ $\quad g^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1/r^2 \end{pmatrix}$

$$\Delta\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\varphi$$

Cov. derivative for $\binom{0}{1}$ tensors:

$$\boxed{p_{\alpha;\beta} \equiv p_{\alpha,\beta} - \Gamma^\mu_{\alpha\beta}\, p_\mu}$$

Higher order tensors:

$$T^{\mu\nu}{}_{;\beta} = T^{\mu\nu}{}_{,\beta} + \Gamma^{\mu}_{\alpha\beta} T^{\alpha\nu} + \Gamma^{\nu}_{\alpha\beta} T^{\mu\alpha}$$

$$A_{\mu\nu;\beta} = A_{\mu\nu,\beta} - \Gamma^{\alpha}_{\mu\beta} A_{\alpha\nu} - \Gamma^{\alpha}_{\nu\beta} A_{\mu\alpha}$$

$$B^{\mu}{}_{\nu;\beta} = B^{\mu}{}_{\nu,\beta} + \Gamma^{\mu}_{\alpha\beta} B^{\alpha}{}_{\nu} - \Gamma^{\alpha}_{\nu\beta} B^{\mu}{}_{\alpha}.$$

Metric & covar derivative

$V^{\alpha}{}_{;\beta} \rightarrow$ tensor $\binom{1}{1}$ $\rightarrow$ $V^{\alpha}{}_{;\beta} = g^{\alpha\nu} V_{\nu;\beta}$

But $V^{\alpha} = g^{\alpha\nu} V_{\nu} \rightarrow (g^{\alpha\nu} V_{\nu})_{;\beta} = g^{\alpha\nu}{}_{;\beta} V_{\nu} +$
$+ g^{\alpha\nu} V_{\nu;\beta} = g^{\alpha\nu} V_{\nu;\beta} \rightarrow \boxed{g^{\alpha\beta}{}_{;\mu} = 0}$

$\rightarrow$ gives relationship between metric & Christoffel

$$\boxed{\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta} \right)}$$

Stress - energy tensor.

Before we move to curved manifolds (non-flat geometry) let us consider a particular tensor field:
stress-energy tensor : $T^{\mu\nu}$ $\binom{2}{0}$-tensor.

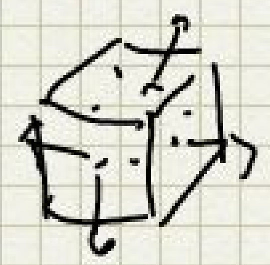
dust

Introduce few notations : the number density $\Rightarrow$
$n = \frac{N}{Vol}$ in MCRF, what number density is for a moving (v) observer? $\Delta V = \Delta x \Delta y \Delta z \rightarrow$

Lorentz contraction $\longrightarrow$ $\Delta V' = \Delta x \Delta y \Delta z \sqrt{1-v^2}$

However the number of "particles" is the same $\rightarrow$

$n' = \dfrac{n}{\sqrt{1-v^2}}$ – number density in frame where particles move (v)



Consider a unit volume and consider a flux of particles through a particular surface

$\Rightarrow$ flux: # of particles / per unit surface / unit time.

In MCRF $\rightarrow$ flux $= 0$

In moving frame (v) $\# = \dfrac{n}{\sqrt{1-v^2}} \cdot \Delta V = \dfrac{n}{\sqrt{1-v^2}} \Delta A \cdot v \Delta t$

$\rightarrow$ flux $= \dfrac{nv}{\sqrt{1-v^2}}$. $\Rightarrow$ generalize $f^i = \dfrac{nv^i}{\sqrt{1-v^2}}$

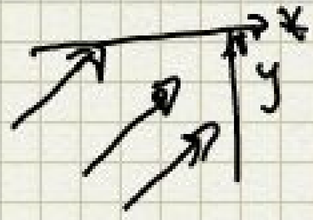
in O' the 4-velocity of particles:

$$u^\alpha\Big|_{O'} = \left\{\frac{1}{\sqrt{1-v^2}}, \frac{\vec{v}}{\sqrt{1-v^2}}\right\}$$

$\rightarrow$ nU^α $\rightarrow$ number flux 4-vector:

$nu^0 \rightarrow$ number density

$nu^i \rightarrow$ flux through the surface in direction "i"



$n_\alpha n^\alpha = -n^2$

Energy density $\rho = n \cdot m$ (in MCRF)

$$\rho\Big|_{O'} = \frac{\rho}{1-v^2}$$

relativistic energy $E^2 - p^2 = m^2$ $(c=1)$

$\rightarrow$

$(c, \vec{v}) \sim (E, \vec{p}) \Rightarrow \quad pc = E\frac{v}{c} \rightarrow$

$E^2 - E^2 v^2 = m^2 \rightarrow E = \frac{m}{\sqrt{1-v^2}}$

$p = \frac{mv}{\sqrt{1-v^2}}$

$\rightarrow \rho\Big|_{O'} = \frac{\rho}{\sqrt{1-v^2}}$ energy density

Stress energy tensor components:

T^{00} - energy density

T^{0i} - energy flux across "i"-th surface

T^{i0} - momentum density

T^{ij} - flux of "i"-th momentum across "j" surface

In MCRF $T^{00} = \rho$, others are zero

For dust $T^{\mu\nu} = p^\mu (n U^\nu) = mn\, U^\mu U^\nu$

$\rightarrow T^{00} = \frac{\rho}{\sqrt{1-v^2}}, \quad T^{0i} = T^{i0} = \frac{\rho v^i}{\sqrt{1-v^2}}, \quad T^{ij} = \frac{\rho v^i v^j}{\sqrt{1-v^2}}$

$\frac{\text{momentum}}{\text{Area} \cdot \text{time}} = \frac{\text{force}}{\text{Area}} \rightarrow$ pressure (stress)

For fluids $\rightarrow$ define macroscopic scalar quantities in MCRF $(T, P, S$ (temper, pressure, spec. entropy)

perfect fluid $T^{\mu\nu} = (\rho + P/c^2)\, u^\mu u^\nu + \frac{P}{c^2} g^{\mu\nu}$

In MCRF $\quad T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}$.

Conservation laws: differential form:

$$\boxed{T^{\mu\nu}{}_{,\nu} = 0}$$

Assume $(nU^\beta)_{,\beta} = 0 \Leftrightarrow \frac{\partial n}{\partial t} + \vec{\nabla}(n\vec{v}) = 0$

$T^{0\alpha}{}_{,\alpha} = 0 \rightarrow$ conservation of energy

$T^{i\alpha}{}_{,\alpha} = 0 \rightarrow$ conservation of momentum

Integral form $\quad \int V^\alpha{}_{,\alpha}\, d^4x = \oint V^\alpha n_\alpha\, d^3S$

$\longleftrightarrow$

Non-flat space-time

2D sphere in 3D is example of a non-flat 2D manifold. We will deal with 4D manifold: space + time. We restrict attention to:

① Differentiable & continuous (can introduce vector field)

② We can define metric on manifold ($g_{\mu\nu}$)

Metric as before is used to measure distance. consider two points infinitesimaly close:

$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \quad \Leftrightarrow \quad \Delta s^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$

Minkowski

 locally flat $\Rightarrow$ how well can approximate $\rightarrow$ depends on curvature

the length $\rightarrow$ $l = \int\limits_{\text{along curve}} ds = \int\limits_{\text{along curve}} \sqrt{g_{\mu\nu}\, dx^\mu dx^\nu}$

Volume element: $dx^0 dx^1 dx^2 dx^3 = \det\left(\Lambda^\alpha{}_{\bar\beta}\right) dx^{\bar 0} dx^{\bar 1} dx^{\bar 2} dx^{\bar 3}$

Jacobian of transformation

define $\det g_{\mu\nu} = g \rightarrow \det\left(\Lambda^\alpha{}_{\bar\beta}\right) = \sqrt{-g}$

$(\det \eta_{\mu\nu} = -1)$ $\rightarrow$ $\underbrace{d^4 V}_{\text{local flat}} = \underbrace{\sqrt{-g}\; d^4 \bar V}$ curved s-t element

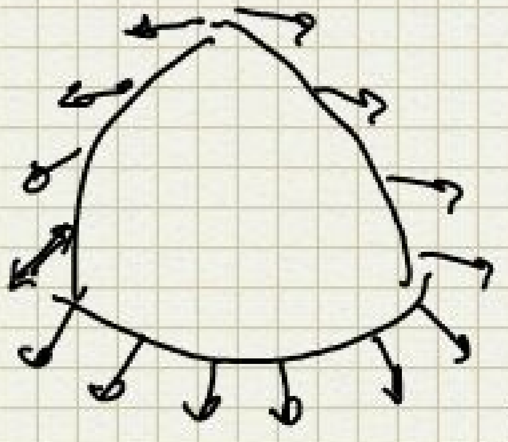
local flatness: metric can be brought by coord. transformation at a point to Minkowski form & Christoffel symbols are $= 0$ at that point.

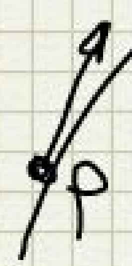
But! $\rightarrow$ can make second derivatives $= 0$ by coord. transformation.

Vector field on manifold

Derivative of vectors involves the difference (comparison) between vectors at two different points (near by).

Parallel transport : transport of a vector along the curve $\Rightarrow$ we preserve the length and vectors parallel in each nearby points





$u^\alpha = \frac{dx^\alpha}{d\lambda} \longrightarrow$ tangent vector

at P in local inertial frame $\frac{dV^\alpha}{d\lambda} = 0$

$$\frac{dV^\alpha}{d\lambda} = u^\beta V^\alpha{}_{,\beta} = u^\beta V^\alpha{}_{;\beta} = 0$$

generalization from localy inertial frame

→ parallel transport of vector V is defined by tangent vector u^α

<u>Geodesics</u> : shortest distance between (curve connecting) two points.

In flat s/t ⇒ straight line. Straight line : parallel remain parallel ⇒ tangent at one point is parallel to tangent at another ⇒ curve parallel transports its own tangent vector

$$\boxed{u^\beta u^\alpha{}_{;\beta} = 0} \qquad u^\beta = \frac{dx^\beta}{d\lambda} \rightarrow u^\beta \frac{\partial}{\partial x^\beta} = \frac{d}{d\lambda}$$

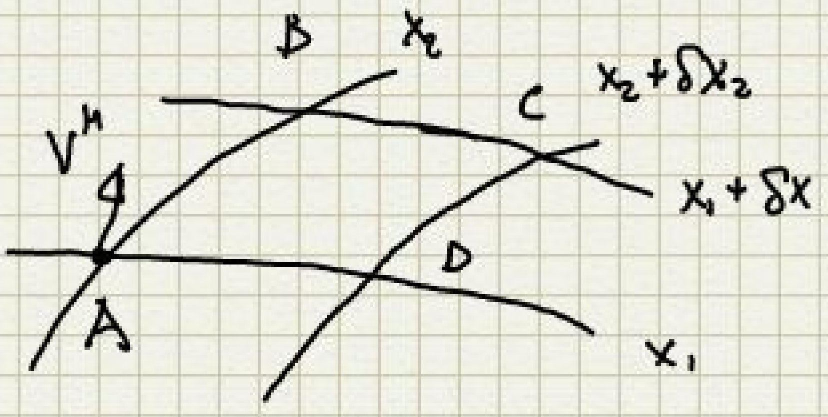
$$\Updownarrow$$

$$\boxed{\frac{d}{d\lambda}\left(\frac{dx^\alpha}{d\lambda}\right) + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0}$$

$\lambda \rightarrow$ affine parameter.

<u>Curvature tensor</u> :

Consider a loop formed by nearby coordinate lines:



make a parallel transport of a vector V^μ : $A \rightarrow B \rightarrow C \rightarrow D \rightarrow A$:

And compute the net change:

$$\delta V^{\mu} = V^{\mu}(A_{final}) - V^{\mu}(A_{init}) \Rightarrow$$

$$\delta V \sim \delta x_1 \, \delta x_2 \, R \, V$$

Riemann curvature tensor

$$R^{\alpha}{}_{\beta\mu\nu} = \Gamma^{\alpha}{}_{\beta\nu,\mu} - \Gamma^{\alpha}{}_{\beta\mu,\nu} + \Gamma^{\alpha}{}_{\rho\mu}\Gamma^{\rho}{}_{\beta\nu} - \Gamma^{\alpha}{}_{\rho\nu}\Gamma^{\rho}{}_{\beta\mu}$$

coming from parallel transport: $U^{\beta} V^{\alpha}{}_{,\beta} = 0$

$U^{\beta} \to$ either x_1 or x_2 (coord. lines). $\Rightarrow$

Riemann curvature tensor $\Rightarrow$

- depends on second deriv. of metric $\Rightarrow$ cannot be eliminated by coord. transformation $\to$ manifest curvature
- In local inertial frame $(\Gamma^{\alpha}{}_{\mu\nu} = 0) \sim$

$$R_{\rho\beta\mu\nu} = g_{\alpha\rho} R^{\alpha}{}_{\beta\mu\nu} = \frac{1}{2}\left[g_{\rho\nu,\beta\mu} - g_{\rho\mu,\beta\nu} + g_{\beta\mu,\rho\nu} - g_{\beta\nu,\rho\mu}\right]$$

$\to$ easy can be seen symmetry properties:

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta}$$

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$$

- $R_{\alpha\beta\mu\nu} = 0 \iff$ flat spacetime
- $V^{\mu}{}_{;\beta;\alpha} - V^{\mu}{}_{;\alpha;\beta} = R^{\mu}{}_{\nu\alpha\beta} V^{\nu}$ cov. derivatives do not commute only flat s/t
- Bianchi identities

$$R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} \equiv 0$$

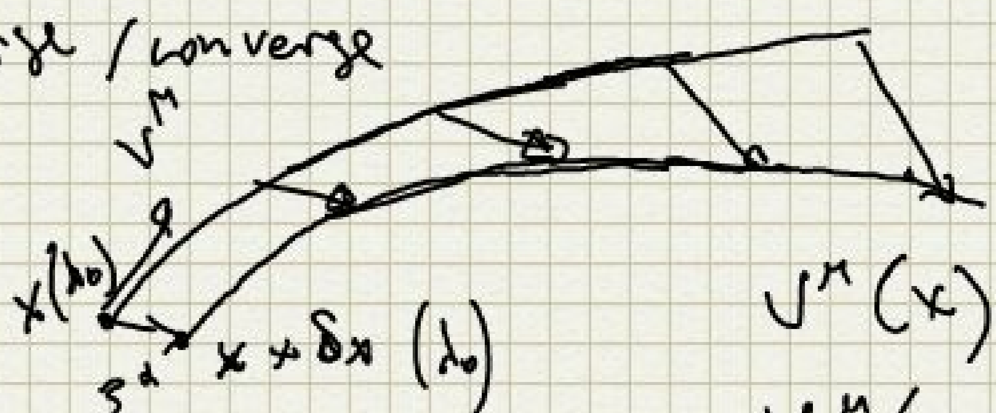
• Ricci tensor $R_{\alpha\beta} \equiv g^{\mu\nu} R_{\mu\alpha\nu\beta} = R^{\mu}{}_{\alpha\mu\beta}$

• Ricci scalar

$$R \equiv g^{\alpha\beta} R_{\alpha\beta} \equiv R$$

Geodesic deviation

Consider two nearby geodesics, due to local flatness ($\delta x \ll R_c$) → remain parallel for some time $\sim \left[\delta x / \frac{\partial x}{\partial \lambda}\right]$ but then if spacetime is not flat the start to diverge / converge



$V^\mu(x)|_{\lambda_0}$ starts parallel to $V^\mu(x+\delta x)|_{\lambda_0}$

$$\rightarrow \boxed{\nabla_V \nabla_V \xi^\alpha = R^\alpha{}_{\mu\nu\beta} V^\mu V^\nu \xi^\beta}$$

∇_V → cov. derivat. along the curve:

$$\nabla_V \xi^\alpha = V^\beta \xi^\alpha{}_{;\beta}$$

Einstein equations

① There are no particles neutral to grav. interaction

Any mass (energy) is affected / creates grav. field

② $m_i a = m_g g$ $\quad a = g$ $\leftrightarrow$ assumes that inertial mass = gravitational charge

$\Rightarrow$ acceleration doesn't depend on mass

$\Rightarrow$ locally inertial frame: freely falling with observer frame $\Rightarrow$ nearby particles have no acceleration $\Rightarrow$

$\Rightarrow$ Freely falling particles move on timelike geodesics

③ Any local physical experiment not involving gravity will have the same result in locally inertial frame (freely falling) & in the flat spacetime

Newtonian gravity

$$\nabla^2 \varphi = 4\pi G \rho \rightarrow \varphi = -\frac{GM}{r}$$: matter is a source

① Want to have Newt. limit

② Independent of coordinate frame

since there is no neutral particles $\Rightarrow$ associate grav. field with non-flat geometry

$f(\partial^2 g) \simeq T$ $\leftarrow$ second order tensor

&

$R^{\mu\nu}$, $g^{\mu\nu} R$, $g^{\mu\nu}$

$T^{\alpha\beta}{}_{;\beta} = 0$ + Newt. limit :

$$\boxed{\underbrace{R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R}_{G^{\alpha\beta}\ \text{Einstein tensor}} + \underbrace{\Lambda g^{\alpha\beta}}_{\Lambda\text{-term}} = \kappa T^{\alpha\beta}} \quad \kappa = 8\pi \ (G = c = 1)$$

$G^{\mu\nu}{}_{;\nu} \equiv 0$ (result of Bianchi identities)

$G^{\mu\nu} = G^{\nu\mu} \to 10 - 4 = 6$ independent components

$\to$ freedom in choosing coordinate frame (analogue of gauge freedom in e/m theory).

in vacuum $\Rightarrow G^{\alpha\beta} = 0 \Leftrightarrow R^{\alpha\beta} = 0$

Consider a weak field limit before going to some exact solutions. $\to$
In absence of gravity $\to$ s/t is flat. Far away from gravitating bodies $\to$ field is weak approaching flat s/t assymptotically (assymptotically flat s/t)
we do not feel attraction from far away galaxies.

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$\eta_{\mu\nu}$ - Minkowski $\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$ in Cartesian coordinates

& $h_{\mu\nu}$ is weak : $|h_{\mu\nu}| \ll 1$

$\to$ Nearly Lorentz coordinates. $\Rightarrow$

$$R_{\alpha\beta\mu\nu} = \frac{1}{2}\left(h_{\alpha\nu,\beta\mu} + h_{\beta\mu,\alpha\nu} - h_{\alpha\mu,\beta\nu} - h_{\beta\nu,\alpha\mu}\right)$$

in linear order in 'h'.

gauge transformation!

$h_{\alpha\beta} \to h^{new}_{\alpha\beta} = h^{old}_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta,\alpha}$, where

ξ_α arbitrary field which respects $|h^{new}_{\alpha\beta}| \ll 1$.

$\to$ keeps $R_{\alpha\beta\mu\nu}(h^{old}) = R_{\alpha\beta\mu\nu}(h^{new}) \to$ gauge invariant

could be associated with coord. transformation:

$$x^{\alpha'} = x^\alpha + \xi^\alpha(x^\beta).$$

Working in linear order in h → indices raised/low with help of background metric (Minkowski in nearly Lorentz frame).

$$h^{\mu}{}_{\beta} = \eta^{\mu\alpha} h_{\alpha\beta}. \qquad h \equiv h^{\alpha}{}_{\alpha} = \eta^{\alpha\beta} h_{\alpha\beta}$$

introduce $\bar{h}^{\alpha\beta} \equiv h^{\alpha\beta} - \frac{1}{2}\eta^{\alpha\beta} h$. →

∃ coordinate frame (gauge) which $\boxed{\bar{h}^{\alpha\beta}{}_{,\beta} = 0}$

Lorentz / harmonic / De Donder gauge

$|T^{00}| \gg |T^{0i}| \gg |T^{ij}|$

$$\to G^{\mu\nu} = 8\pi T^{\mu\nu} \longrightarrow \Box \bar{h}^{00} = -16\pi\rho$$ (lowest order in v, h)

$$\Box \bar{h}^{00} = \eta^{\alpha\beta} \bar{h}^{00}{}_{,\alpha,\beta} \quad , \quad \Box = \nabla^2 - \partial^2/\partial t^2 \sim \left(v \frac{\partial}{\partial x}\right)^2$$

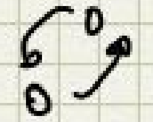
$$\to \nabla^2 \bar{h}^{00} = -16\pi\rho \Longleftrightarrow \nabla^2 \Phi = 4\pi\rho$$ (Newton's equation)

$$\to \bar{h}^{00} = -4\Phi \to h^{00} = -2\Phi = h^{xx} = h^{yy} = h^{zz}$$

$$\to ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2+dy^2+dz^2)$$

$$\Phi = -M/r$$

Propagation of GW:

 $\xrightarrow[\text{away}]{\text{far, far}}$ $\quad \Box G^{\mu\nu} = 0 \qquad g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

$$\Box \bar{h}^{\alpha\beta} = 0 \to A^{\alpha\beta} \exp^{i k_\nu x^\nu}$$ (solution)

(plane monochromatic wave)

$$\to \eta^{\mu\nu} k_\mu k_\nu = k_\mu k^\mu = 0$$ → null

$$k^\mu \doteq (-\omega^2, \vec{k}) \to \omega^2 = |\vec{k}|^2$$ → propagates with c

$$\bar{h}^{\alpha\beta}{}_{,\beta} = 0 \to A^{\alpha\beta} k_\beta = 0$$ → transverse

We can apply gauge transformation

$\left[\bar{h}^{\alpha\beta}{}_{,\beta} = 0\right.$ → harmonic gauge is not unique ⇒ further freedom $\Box \xi^\alpha = 0$ → can be added preserves

$\Box \bar{h}^{\alpha\beta} = 0$ & $\bar{h}^{\alpha\beta}{}_{,\beta} = 0$ → can choose so that:

$A^\alpha{}_\alpha = 0$ $\qquad A_{\alpha\beta} u^\beta = 0$ $\quad$ (u^β any timelike vector)

traceless $\qquad u^\beta = \{u^0, 0, 0, 0\} \rightarrow A^{0\beta} = 0$

→ GW ⇒ can choose the gauge where it is transvers and traceless and $\bar{h}^{0i} = 0$

We will consider GW further during second IMPRS week.

<u>Schwarzschild solution</u>

We search for solution of Einstein eqns. which

(i) static (does not depend on time)

(ii) spherically symmetric

(iii) vacuum solution $\quad R^{\mu\nu} = 0$

Use spher. coord

$$ds^2 = -g_{00}\, dt^2 + 2 g_{0r}\, dr\, dt + g_{rr}\, dr^2 + r^2 \underbrace{d\Omega^2}_{d\theta^2 + \sin^2\theta\, d\varphi^2}$$

r → circumference $2\pi r$

static implies symmetry $t \to -t$ ⇒ $g_{0r} = 0$

assymptotically flat : $g_{\mu\nu} \to \eta_{\mu\nu}$ as $r \to \infty$

G_{00}, G_{rr}, $G_{AA} = 0$ → non-trivial

Solution

$$ds^2 = -\left(1-\frac{r_g}{r}\right)dt^2 + \left(1-\frac{r_g}{r}\right)^{-1}dr^2 + r^2 d\Omega^2$$

far away we want to recover Newtonian limit

$$g_{00} \underset{r\to\infty}{\rightarrow} \eta + 2\phi \rightarrow \phi = \frac{M}{r} \rightarrow \boxed{r_g = 2M}$$

$$ds^2 \approx -\left(1-\frac{2M}{r}\right)dt^2 + \left(1+\frac{2M}{r}\right)dr^2 + r^2 d\Omega^2 \quad \text{far away}$$

$r_g \rightarrow$ grav. radius of a body or Schwarzschild radius (in spherical, Schwarzschild coord.)

Orbits in Schwarzschild s/t.

Consider local conservation law $T^{\mu\nu}{}_{;\nu} = 0$

$\rightarrow$ geodesic equation for a test particle $\rightarrow$

$$u^\beta u^\alpha{}_{;\beta} = 0 \qquad u^\alpha - \text{4-velocity}$$

Use another affine parameter $\tau \rightarrow \tau/m$, m is a const. which we associate with a mass:

$$p_\mu p^\mu = -m^2 \qquad p^\mu - \text{4-momentum}$$

$\Rightarrow$ $p^\beta p^\alpha{}_{;\beta} = 0$ (another representation of a geodesic).

use metric expression

$$\rightarrow m\frac{dp_\beta}{d\tau} = \Gamma^\nu{}_{\beta\alpha} p^\alpha p_\nu = \frac{1}{2} g_{\nu\alpha,\beta} p^\nu p^\alpha$$

$\rightarrow$ if $g_{\alpha\beta}$ doesn't depend on q (one of the coord: x^0, x^1, x^2, x^3) $\rightarrow$ $q \overset{\text{say}}{=} x_i$ $\rightarrow$ $\frac{dp_i}{d\tau} = 0 \rightarrow p_i$ is conserved along geodesic

Newtonian limit

$$p^0 \approx m\left(1 - \phi + \frac{|\vec{p}|^2}{2m^2}\right), \quad -p_0 = m + \underbrace{m\phi + |\vec{p}|^2/2m}_{\text{potential \& kinetic energy}}$$

In Schwarzschild s/t $g_{\mu\nu} = g_{\mu\nu}(r) \to$ independent of $t, \varphi, \theta \to$

p_0, p_φ, p_θ are conserved $\to$ Energy & angular momentum

$\tilde{E} = -\frac{p_0}{m}$ (particle), $E = -p_0$ (photon)

$\tilde{L} = \frac{p_\varphi}{m}$ (particle), $L = p_\varphi$ (photon)

can always choose $\theta = \pi/2$ & orbit confined to a plane.

Particle:

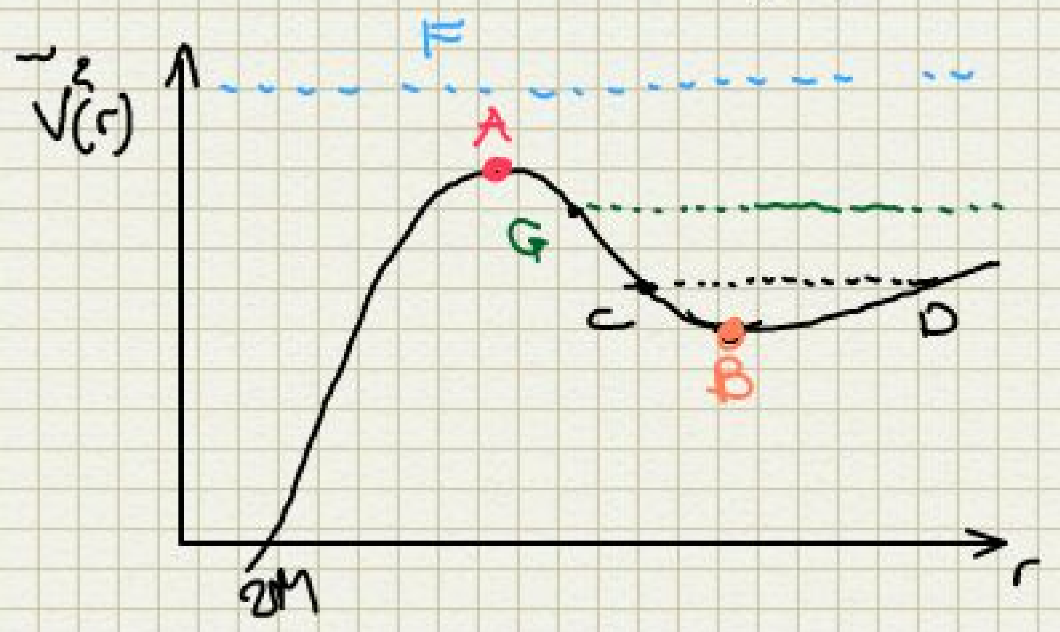
$$\underbrace{\begin{aligned} p^0 &= g^{00} p_0 \\ p^r &= g^{rr} p_r = m \frac{dr}{d\tau} \\ p^\varphi &= g^{\varphi\varphi} p_\varphi \end{aligned}}_{\text{particle}}$$

$g_{\mu\nu} p^\mu p^\nu = -m^2$ (super Hamiltonian)

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \tilde{V}^2(r)$$

$$\tilde{V}^2(r) = \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\tilde{L}^2}{r^2}\right)$$

$\tilde{E}^2 > \tilde{V}^2$



Photon:

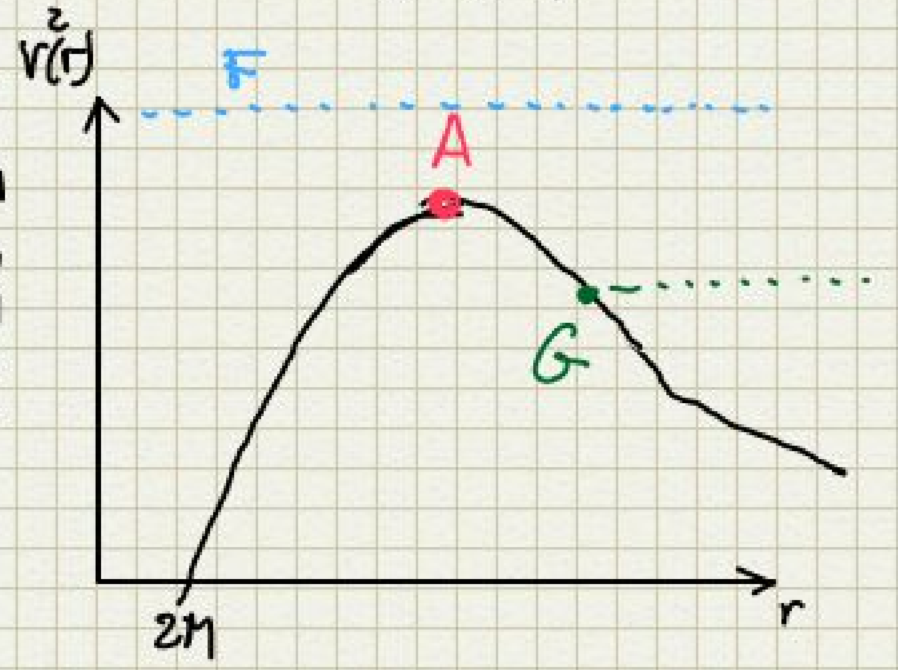
$$\underbrace{\begin{aligned} p^0 &= g^{00} p_0 \\ p^r &= \frac{dr}{d\lambda} \\ p^\varphi &= \frac{d\varphi}{d\lambda} = \frac{L}{r^2} \end{aligned}}_{\text{photon}}$$

$g_{\mu\nu} p^\mu p^\nu = 0$ (null)

$$\left(\frac{dr}{d\lambda}\right)^2 = E^2 - V^2(r)$$

$$V^2(r) = \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2}$$

$E^2 > V^2$



$\boxed{G}$ part/photon comes from ∞ reaches distance G & returns back to ∞ (G : turning point $E^2 = V^2$)

$$\frac{d^2r}{d\tau^2} = -\frac{1}{2}\frac{d}{dr}\tilde{V}^2 \qquad \frac{d^2r}{d\lambda^2} = -\frac{1}{2}\frac{d}{dr}V^2$$

coming particle has acceleration outwards, G → turning point $\frac{dr}{d\tau} = \frac{dr}{d\lambda} = 0$ ⇒ hyperbolic orbit

$\boxed{B}$ minimum : $r = const$ → circular orbit

$$\frac{d^2r}{d\tau^2} \approx \frac{d\tilde{V}}{dr} = 0 \rightarrow \text{zero radial acceleration.}$$

→ circular orbit.

$\boxed{A}$ → another extremum (maximum) → unstable point: small change in r + goes either inwards (plunge like F) or outwards ⇒ ∃ for particles & photons

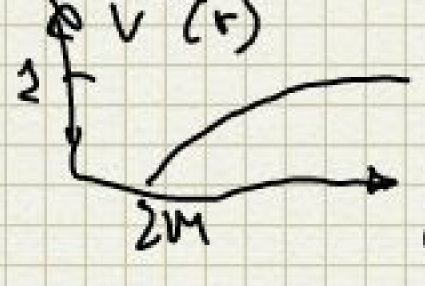
$$r_{A,B} = \frac{\tilde{L}^2}{2M}\left[1 \pm \sqrt{1 - \frac{12M^2}{\tilde{L}^2}}\right]$$

$\tilde{L}^2 > 12M^2$

$\tilde{L}^2 = 12M^2 \rightarrow r_A = r_B = r_{min} = 6M$

last stable circular orbit

$\tilde{L}^2 < 12M^2$



for photons

$r_A = 3M$

→ light ring

$\boxed{F}$ direct plunge goes through $r = 2M$ & never comes back (small impact parameter)

Here we assumed that the size of the mass M is less than Schwarzschild radius $r_g = 2M$ ⇒

⇒ Black Hole (BH)

Let's look at φ - motion $\Rightarrow$

$$\left.\begin{aligned} \frac{d\varphi}{d\tau} &= \frac{p^\varphi}{m} = g^{\varphi\varphi}\tilde{L} = \frac{1}{r^2}\tilde{L} \\ \frac{dt}{d\tau} &= \frac{p^0}{m} = g^{00}\frac{p_0}{m} = g^{00}(-\tilde{E}) = \frac{\tilde{E}}{1-2M/r} \end{aligned}\right\} \Rightarrow$$

$\frac{d\varphi}{dt} = \left(\frac{r^3}{M}\right)^{-1/2}$: coordinate period (time to change φ by 2π) $\Rightarrow$ $P = 2\pi\sqrt{r^3/M}$

$\boxed{\text{perihelion precession}}$

test mass : C $\leftrightarrow$ D $\overset{\circ}{\circ}$ eccentric motion r-coord oscillates between r_{min} (C) & r_{max} (D)

Period of radial motion : time to travel $r_C \to r_D \to r_C$

Does not coincide with φ-period :

$\Delta\varphi = \Delta\varphi(T_r)$:

$$\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - \tilde{V}^2 \; ; \quad \frac{d\varphi}{d\tau} = \frac{\tilde{L}}{r^2} \Rightarrow$$

$$\left(\frac{dr}{d\varphi}\right)^2 = \frac{\tilde{E}^2 - \left(1-\frac{2M}{r}\right)\left(1+\tilde{L}^2/r^2\right)}{\tilde{L}^2/r^4}$$

consider nearly circular orbit $\Rightarrow$

$$\Delta\varphi = 2\pi\left(1 - 6M^2/\tilde{L}^2\right)^{-1/2}$$ change in φ as test mass moves $r_C \to r_D \to r_C$

$$\Delta\varphi \simeq 2\pi\left(1 + 3M^2/\tilde{L}^2\right)$$ (for nearly Newtonian orbit)

(Newtonian $\frac{1}{r} \simeq M/\tilde{L}^2 \leftrightarrow$ circ. orbit $\tilde{L}^2 = \frac{Mr}{1-3M/r} \simeq Mr$)

$\Rightarrow$ precession of perihelion (periapse)

$\Delta\varphi_{adv} = 6\pi M^2/\tilde{L}^2$ (rad per orbit)

$\approx 6\pi\, M/r$

$\Delta\varphi^{adv}_{Mercury} \approx 5\times10^{-7}$ rad/orbit

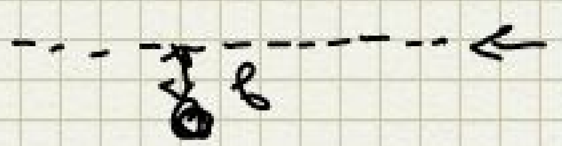
$\Delta\varphi^{adv}_{H-T} \approx 3.3\times10^{-5}$ rad/orbit ($\approx 2^\circ$/year) ⇐ Hulse-Taylor system.

Photon's azimuthal motion

(similar to above, but for the photon) ⇒

$$\frac{d\varphi}{dr} = \pm \frac{1}{r^2\left(\frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{2M}{r}\right)\right)^{1/2}}$$

b → impact parameter $b = \frac{L}{E}$ → Newt. analogue ⇒



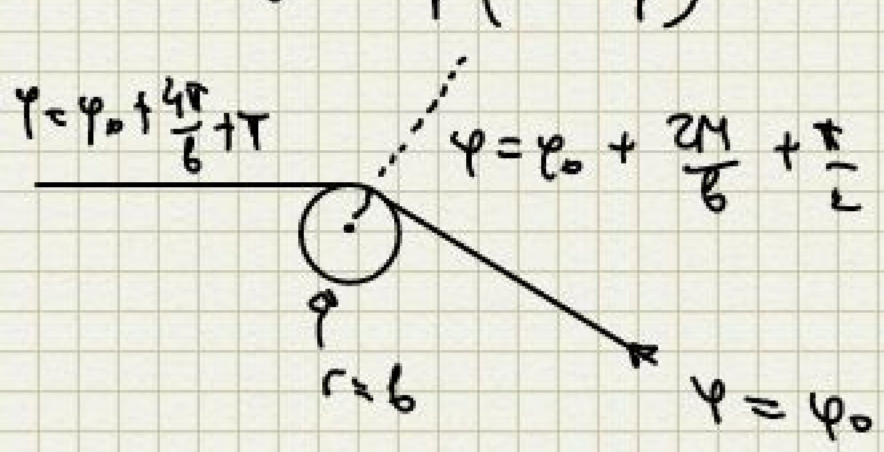
Assume $M/r \ll 1$ and work up to $(M/r)^3$

In Newtonian theory $\varphi = \varphi_0 + \arcsin\frac{b}{r}$ (straight line)

In Schwarzschild geometry

$$\varphi = \varphi_0 + \frac{2M}{b} + \arcsin(by) - 2M\left(\frac{1}{b^2} - y^2\right)^{1/2}$$

$$y = \frac{1}{r}\left(1 - \frac{M}{r}\right)$$



$\frac{dr}{d\lambda} = 0$ at $r = b$ (min approach)

$$\varphi = \varphi_0 + \frac{2M}{b} + \frac{\pi}{2}$$

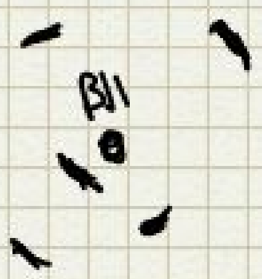
$$\boxed{\Delta\varphi = \frac{4M}{b}}$$ deflection of light

for sun: $\Delta\varphi = 8.45\times10^{-6}$ rad ($b = R_\odot \approx 7\times10^5$ km)

Gravitational lensing

(Einstein ring)

usually observe several images of the same object and or images are streches in one direction & squeezed in another



Properties of $r = r_g = 2M$ surface

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

(Schwarzschild coordinates)

$$\left.\begin{array}{ll} g_{00} \to 0 & \text{as } r \to r_g \\ g_{rr} \to \infty & \text{as } r \to r_g \end{array}\right\} \text{ something strange here}$$

let's drop an astronaut directly into BH to investigate

$\tau \to$ proper time: measured by astronaught

Use eqn $\left(\frac{dr}{d\tau}\right)^2 = \tilde{E}^2 - 1 + \frac{2M}{r}$ (L=0 radial motion)

$\tilde{E} = 1 \to$ falling from ∞ (where $V=0$)

$$\Delta\tau = \frac{4M}{3}\left[\left(\frac{r}{2M}\right)^{3/2}\right]_{2M}^{R} \quad \text{(from R to 2M)}$$

everything is ok

look at the coordinate time measured by us (at ∞) after we sent astronaut

$$dt = \frac{-(\varepsilon + 2M)^{3/2}}{\sqrt{2M}\,\varepsilon} d\varepsilon \quad ; \varepsilon = r - 2M \; ; \; t \to \infty \text{ as } r \to 2M$$

so for us the astronaut never reaches the $r=2M$ surface.

→ bad coordinates ⇒ what happens at $2M$ & beyond?

We can compute coord invariant quantities $R^{\alpha}{}_{\mu\nu\beta}$, $R_{\alpha\beta}$, R → all are finite at $r=2M$ → again indicates that this is a problem with coordinates. $r<2M$ ⇒

$$g_{00}\,dt^2 = -\left(1-\frac{2M}{r}\right) > 0 \rightarrow \text{space-like behaviour}$$

$$g_{rr}\,dr^2 = \left(1-\frac{2M}{r}\right)^{-1} < 0 \rightarrow \text{time-like behaviour}$$

$r=0$ → true singularity ($R^{\alpha}{}_{\beta\mu\nu} \Rightarrow \infty$)

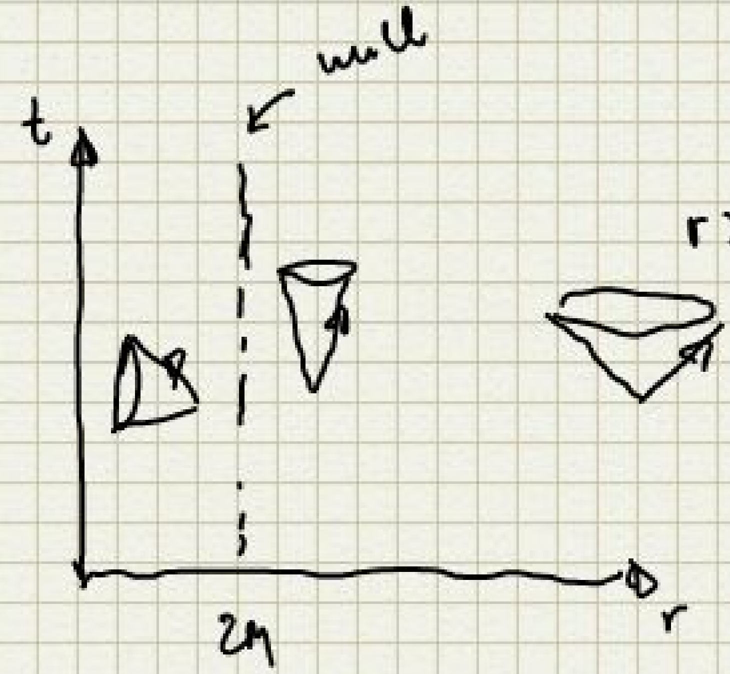
Let us send photon

$$ds^2 = 0 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2$$

$$\rightarrow \frac{dt}{dr} = \pm\left(\frac{1}{1-\frac{2M}{r}}\right) \quad \text{outgoing}$$

$r>2M$ $\frac{dt}{dr}>0$

$r<2M$ $\frac{dt}{dr}<0$ (ingoing)



outgoing photon cannot escape at ∞ from underneath $r=2M$ surface

If particle or light crosses $r=2M$ surface → it cannot escape (cannot be seen by external observer) → horizon (physical)

Is there better coordinate frame?

Kruskal - Szekers coordinates (only 1960!)

$$u = \sqrt{\pm\left(r/2M - 1\right)}\; e^{r/4M} \quad \cosh\left(t/4M\right) \quad \sinh\left(t/4M\right)$$

$$v = \sqrt{\pm\left(r/2M - 1\right)}\; e^{r/4M} \quad \sinh\left(t/4M\right) \quad \cosh\left(t/4M\right)$$

— $r > 2M$

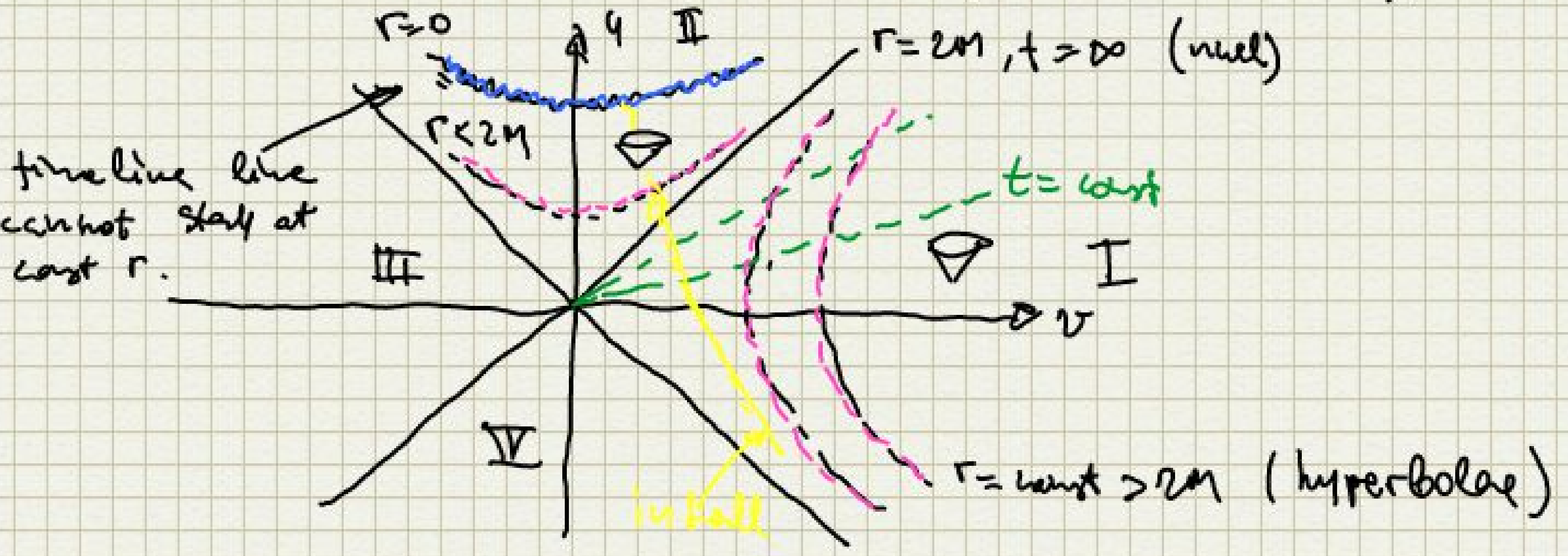
— $r < 2M$

Transformation is singular at $r = 2M$ (removes artif. coord. singularity at $r = 2M$).

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}\left(dv^2 - du^2\right) + r^2 d\Omega^2$$

Here $r = r(u,v)$: $u^2 - v^2 = \left(\frac{r}{2M} - 1\right) e^{r/2M}$

- regular at $r = 2M$ (singularity at $r = 0$)
- radial null: $dv^2 = du^2$ (similar to SR)



Region I → exterior of BH, II → interior

If the size of object is > 2M: need to consider non-vacuum solution $G^{\mu\nu} = \kappa T^{\mu\nu}$ & get metric inside the star.

Kerr BH

→ static & axially symmetric solution of vacuum Einstein equations $G^{\mu\nu} = 0$

→ defined by two parameters: M, J or $a = J/M$ $[m]$ $[m^2]$

$$ds^2 = -\frac{\Delta - a^2 \sin^2\theta}{\rho^2} dt^2 - 2a\frac{2Mr\sin^2\theta}{\rho^2} dt\, d\varphi +$$

$$+ \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\, d\varphi^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

$$\Delta = r^2 - 2Mr + a^2 \; ; \quad \rho^2 = r^2 + a^2\cos^2\theta$$

$t, r, \theta, \varphi,$ → here Boyer - Lindquist coord.

$r \to \infty$ → oblate spheroidal coord. (try it).

① $t = $ const, $r = $ const not sphere

② $a = 0$ → reduces to Schwarzschild metric

③ $g_{t\varphi} \neq 0 = -a\frac{2Mr\sin^2\theta}{\rho^2}$

④ metric doesn't depend on t, φ → at least two conserved quant.

p_φ, p_t → L_z, E : projection of angular momentum on spin of BH, energy

There is third constant: Carter constant Q (or C)

Consider ③ & ④ Have a test mass with zero angular momentum: $p_\varphi = 0$

But $\frac{d\varphi}{dt} = \frac{p^\varphi}{p^t} \simeq \frac{g^{\varphi\alpha} p_\alpha}{g^{t\alpha} p_\alpha} = \frac{g^{\varphi t} p_t}{g^{tt} p_t} = \frac{g^{\varphi t}}{g^{tt}} \doteq \omega(r,\theta)$

(angular vel.)

non zero

$$g^{t\varphi} = -a \frac{2Mr}{\rho^2 \Delta} \quad ; \quad g^{tt} = -\frac{(r^2+a^2)^2 - a^2 \Delta \sin^2\theta}{\rho^2 \Delta}$$

so the astronaut dropped radially into Kerr BH will acquire angular velocity : BH rotation drags it using gravity $\omega \sim 1/r^3$ ⇒ dragging of inertial frame

⑤ $g_{tt} = 0$ not a horizon

Consider a photon in equatorial plane $(\theta = \pi/2)$ at r, tangent to a circle at $r =$ const

$\rightarrow ds^2 = 0 = g_{tt}\, dt^2 + 2 g_{t\varphi}\, dt\, d\varphi + g_{\varphi\varphi}\, d\varphi^2$ if $g_{tt} = 0$

$\rightarrow \frac{d\varphi}{dt} = 0$ and $\frac{d\varphi}{dt} = -\frac{2 g_{t\varphi}}{g_{\varphi\varphi}} \sim \text{sign}(a)$

photon doesn't move

(sent in opposite to rotation dir) ← extreme dragging

$g_{tt} = 0 \rightarrow$ ergosphere (static limit) : no particle can remain at rest inside ergosphere

$r_{ergosph.} = M + \sqrt{(M^2 - a^2 \cos^2\theta)}$

⑥ Horizon → exist at $g_{rr} \to \infty$: $\Delta = 0$

$r_+ = r_{hor} = M + \sqrt{M^2 - a^2} \leq 2M$

consider $t =$ const $r =$ const → surface (θ, φ)

$$dl^2 = \frac{(r^2+a^2)^2 - a^2\Delta}{\rho^2} \sin^2\theta \, d\varphi^2 + \rho^2 d\theta^2 \quad (dt = dr = 0)$$

$$\equiv \gamma_{ij}\, d\xi^i d\xi^j \quad ; \quad \xi^i = \{\theta, \varphi\} \; ; \; \gamma_{ij} \to \text{metric on}$$

the surface $\Rightarrow$

$$\text{Area}: \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sqrt{\det(\gamma_{ij})} = 4\pi \sqrt{(r^2+a^2)^2 - a^2\Delta}$$

for horizon : $\Delta = 0 \to$ $A_{hor} = 4\pi (\underbrace{r_+^2}_{r_{horizon}} + a^2)$.

<u>Properties of BHs</u>

① Isolated BH should become stationary.

No infalling matter. characterized by <u>only 2</u> parameters, M, a

② If BH not in vacuum $\to$ above doesn't hold

③ Consider collapsing star or throw some matter onto BH (non-spherical) $\to$ resulting BH is axy-symmetric : all deviations (quadrupole moment octupole moment, ...) are radiated away: excited BH formed $\to$ excitations are radiated by GW (Quasinormal modes) $\to$ Kerr or Sch BH is left after some time.

④ In any dynamical process (accretion of gas) the total area of horizon cannot decrease (assumes that local energy density of matter $\rho > 0$

This implies that BH formed by merger of two BHs or NSs cannot bifurcate spontaneously into two smaller ones.

⑤ $r=0$ Singularity $R^{\alpha}{}_{\beta\mu\nu}=\infty$: can be seen as gravity becomes strong enough for breaking GR (classical) $\rightarrow$ probably need quantum gravity

In BHs singularity is hidden inside horizon.

Penrose $\Rightarrow$ cosmic censorship conjecture : no naked singularity can arise out of non-singular initial conditions in asymptotically flat s/t